

Strings and branes in plane waves

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Abstract: An overview of string theory in the maximally supersymmetric plane-wave background is given, and some supersymmetric D-branes are discussed.

1 Introduction & Motivation

According to the Maldacena conjecture [1] type IIB string theory on $\text{AdS}_5 \times \text{S}^5$ is dual to four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $\text{SU}(N)$, where the parameters of the two theories are related as

$$\begin{aligned} g_s &= g^2, \\ R &= (g^2 N)^{1/4}. \end{aligned} \tag{1}$$

Here g_s is the string coupling constant, while g denotes the coupling constant of the Yang-Mills theory. The second equation implies that the radius of $\text{AdS}_5 \times \text{S}^5$ is proportional to a power of the t'Hooft parameter $x = g^2 N$. In the large N limit for fixed but large t'Hooft parameter x , the string theory is weakly coupled and the curvature terms are small. We can then use the supergravity approximation to string theory to calculate quantities on the string theory side, and compare them with Yang-Mills theory. This is the limit in which most tests of the Maldacena conjecture have been performed. However, the above duality is meant to hold not only in this limit, but also as a duality at *finite* N , relating a string theory with a four-dimensional conformal field theory. Unfortunately, string theory on $\text{AdS}_5 \times \text{S}^5$ is so far not solvable, and it is therefore difficult to check this conjecture directly.

Recently it was observed that there exists a limit in which the string theory becomes exactly solvable. This is the so-called *Penrose limit* [2, 3, 4, 5, 6]. One can take a corresponding limit in the dual gauge theory, and this then allows for a quantitative analysis of at least some aspects of the Maldacena correspondence. This line of thought has been successfully pursued in a number of papers (see for example [7, 8, 9, 10] for a very incomplete list of papers, and [11, 12, 13, 14] for some recent reviews). These developments will be described by other speakers; here we shall concentrate on explaining how the string theory can be exactly solved in this background.

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To begin with we need to explain how to construct the Penrose limit in detail. To this end recall that the metric of $\text{AdS}_5 \times \text{S}^5$ can be written as

$$ds^2 = R^2 \left[-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + \cos^2 \theta d\psi^2 + d\theta^2 + \sin^2 \theta d\Omega_3'^2 \right], \quad (2)$$

where the first three terms describe the metric of AdS_5 , while the last three terms describe that of S^5 . Let us consider a null geodesic that is defined by

$$\rho = 0, \quad \theta = 0, \quad \psi = t. \quad (3)$$

The Penrose limit is simply the space that is obtained by blowing up a small neighbourhood of this null geodesic. To this end introduce light-cone coordinates $\tilde{x}^\pm = \pm(t \pm \psi)/\sqrt{2}$, and rescale the coordinates by defining

$$x^+ = \tilde{x}^+, \quad x^- = R^2 \tilde{x}^-, \quad \rho = \frac{r}{R}, \quad \theta = \frac{y}{R}, \quad (4)$$

and taking $R \rightarrow \infty$. It is an easy exercise to check that the above metric becomes in this limit

$$ds^2 = 2 dx^+ dx^- - \frac{1}{4} (r^2 + y^2) dx^+ dx^+ + d\mathbf{y}^2 + d\mathbf{r}^2, \quad (5)$$

where \mathbf{r} and \mathbf{y} are four-dimensional vectors whose length is r and y , and whose angular directions are described by the three-spheres of AdS_5 and S^5 , respectively. [In the following we shall also sometimes denote the four coordinates \mathbf{r} that come from AdS_5 by x^1, \dots, x^4 , and the four coordinates \mathbf{y} that come from S^5 by x^5, \dots, x^8 .] In addition, $\text{AdS}_5 \times \text{S}^5$ has a non-trivial RR 5-form field strength, and in the above limit only the

$$F_{+1234} = F_{+5678} = 2\mu \quad (6)$$

components survive. With this normalisation the above metric is then

$$ds^2 = 2dx^+ dx^- - \mu^2 x^I x^I dx^+ dx^+ + dx^I dx^I, \quad (7)$$

where I runs from $I = 1, \dots, 8$. This is the metric of a *plane-wave background*. [A general plane-wave background is of the form

$$ds^2 = 2dx^+ dx^- + A_{IJ}(x^+) x^I x^J dx^+ dx^+ + dx^I dx^I; \quad (8)$$

the above is the special case for which $A_{IJ}(x^+) = -\mu^2 \delta_{IJ}$.] In the following we shall concentrate on describing string theory in this special case; it has now been realised that more general backgrounds also lead to exactly solvable string theories (see for example [15, 16, 17]). There are also some related backgrounds with NS-NS flux that can be studied in detail (see for example [18]).

2 String theory in the plane wave background

It has been known for some time that plane wave backgrounds are exact solutions of string theory [19, 20]. However, it has only recently been realised that string theory is exactly solvable in these backgrounds in the GS formalism in light-cone gauge [3, 4]. By varying the sigma-model action with respect to x^- , one finds that x^+ satisfies the usual wave equation, which can therefore be solved by setting

$$x^+ = 2\pi\alpha' p^+ \tau, \quad (9)$$

where τ is the world-sheet time. In light-cone gauge, x^- is determined by the constraint equations in terms of the transverse coordinates (and p^+), and the Lagrangian for the transverse coordinates becomes (with $\alpha' = 1$)

$$\mathcal{L} = \frac{1}{4\pi} \left(\partial_+ x^I \partial_- x^I - m^2 (x^I)^2 \right) + \frac{i}{2\pi} \left(S^a \partial_+ S^a + \tilde{S}^a \partial_- \tilde{S}^a - 2m S^a \Pi_{ab} \tilde{S}^b \right). \quad (10)$$

Here m is defined by $m = 2\pi p^+ \mu$, and S^a and \tilde{S}^a are $SO(8)$ spinors of the same chirality. The matrix Π is given as $\Pi = \gamma^1 \gamma^2 \gamma^3 \gamma^4$, where the 8×8 matrices, γ_{ab}^I and $\gamma_{\dot{a}\dot{b}}^I$, are the off-diagonal blocks of the 16×16 $SO(8)$ γ -matrices and couple $SO(8)$ spinors of opposite chirality. The presence of Π in the fermionic sector of the Lagrangian (that reflects the RR 5-form flux) breaks the symmetry from $SO(8)$ to $SO(4) \times SO(4)$.

This Lagrangian describes a field theory of eight free massive bosons and fermions. The mass term implies, in particular, that the theory is not conformally invariant in light cone gauge.¹ However, it is still a free theory, and we can therefore solve it explicitly. For example, the equations of motion for the transverse bosons imply that we have the mode expansion

$$x^I(\sigma, \tau) = \cos(m\tau) x_0^I + \frac{\sin(m\tau)}{m} p_0^I + i \sum_{n \neq 0} \frac{1}{\omega_n} \left(e^{-i\omega_n \tau + 2\pi n i \sigma} \alpha_n^I + e^{-i\omega_n \tau - 2\pi n i \sigma} \tilde{\alpha}_n^I \right), \quad (11)$$

where

$$\omega_n = \text{sign}(n) \sqrt{m^2 + n^2}. \quad (12)$$

Upon quantisation this leads to the commutation relations

$$[\alpha_k^I, \alpha_l^J] = \omega_k \delta^{IJ} \delta_{k,-l}, \quad [\alpha_k^I, \tilde{\alpha}_l^J] = 0, \quad [\tilde{\alpha}_k^I, \tilde{\alpha}_l^J] = \omega_k \delta^{IJ} \delta_{k,-l}. \quad (13)$$

In addition there are the bosonic zero modes that describe the centre of mass position x_0^I and some generalised momentum p_0^I with $[p_0^I, x_0^J] = -i\delta^{IJ}$. It is convenient to introduce the creation and annihilation operators

$$a_0^I = \frac{1}{\sqrt{2m}} (p_0^I + i m x_0^I), \quad \bar{a}_0^I = \frac{1}{\sqrt{2m}} (p_0^I - i m x_0^I), \quad (14)$$

in terms of which the commutation relations are then $[\bar{a}_0^I, a_0^J] = \delta^{IJ}$. These are the familiar commutation relations of a harmonic oscillator; this was to be expected as the zero modes describe a point particle in a harmonic oscillator potential. Similarly we can expand the fermionic fields in terms of modes S_k^a and \tilde{S}_k^a , where a is a spinor index of $SO(8)$ and $k \in \mathbb{Z}$. These modes then satisfy the anti-commutation relations

$$\{S_k^a, S_l^b\} = \delta^{ab} \delta_{k,-l}, \quad \{S_k^a, \tilde{S}_l^b\} = 0, \quad \{\tilde{S}_k^a, \tilde{S}_l^b\} = \delta^{ab} \delta_{k,-l}. \quad (15)$$

For future convenience we also introduce the zero-mode combinations $\theta_0^a = \frac{1}{\sqrt{2}}(S_0^a + i\tilde{S}_0^a)$ as well as $\bar{\theta}_0^a = \frac{1}{\sqrt{2}}(S_0^a - i\tilde{S}_0^a)$, and further

$$\begin{aligned} \theta_R &= \frac{1}{2}(1 + \Pi)\theta_0, & \bar{\theta}_R &= \frac{1}{2}(1 + \Pi)\bar{\theta}_0, \\ \theta_L &= \frac{1}{2}(1 - \Pi)\theta_0, & \bar{\theta}_L &= \frac{1}{2}(1 - \Pi)\bar{\theta}_0. \end{aligned} \quad (16)$$

¹It is believed [21] that the covariant theory is conformally invariant. By going to light-cone gauge we have used up the conformal symmetry, and there is *a priori* no reason why the resulting theory should still be conformally invariant. The fact that for flat Minkowski space the theory in light-cone gauge is still conformally invariant is a result of some additional symmetries of the Minkowski background.

The theory is maximally supersymmetric: in light-cone gauge the 32 supercharges of type IIB string theory decompose into sixteen ‘kinematical’ supercharges that transform in the undotted spinor representation of $SO(8)$, and sixteen ‘dynamical’ supercharges that transform in the dotted representation. The kinematical supercharges are simply given by $Q_a \equiv S_0^a$ and $\tilde{Q}_a \equiv \tilde{S}_0^a$, while the formula for the dynamical supercharges is [4, 22]²

$$\begin{aligned} \sqrt{2p^+} Q_{\dot{a}} &= p_0^I \gamma_{\dot{a}b}^I S_0^b - m x_0^I (\gamma^I \Pi)_{\dot{a}b} \tilde{S}_0^b \\ &+ \sum_{n=1}^{\infty} \left(c_n \gamma_{\dot{a}b}^I (\alpha_{-n}^I S_n^b + \alpha_n^I S_{-n}^b) + \frac{im}{2\omega_n c_n} (\gamma^I \Pi)_{\dot{a}b} (\tilde{\alpha}_{-n}^I \tilde{S}_n^b - \tilde{\alpha}_n^I \tilde{S}_{-n}^b) \right), \end{aligned} \quad (17)$$

$$\begin{aligned} \sqrt{2p^+} \tilde{Q}_{\dot{a}} &= p_0^I \gamma_{\dot{a}b}^I \tilde{S}_0^b + m x_0^I (\gamma^I \Pi)_{\dot{a}b} S_0^b \\ &+ \sum_{n=1}^{\infty} \left((c_n \gamma_{\dot{a}b}^I (\tilde{\alpha}_{-n}^I \tilde{S}_n^b + \tilde{\alpha}_n^I \tilde{S}_{-n}^b) - \frac{im}{2\omega_n c_n} (\gamma^I \Pi)_{\dot{a}b} (\alpha_{-n}^I S_n^b - \alpha_n^I S_{-n}^b) \right), \end{aligned} \quad (18)$$

where c_n is defined by

$$c_n = \frac{m}{\sqrt{2\omega_n(\omega_n - n)}}. \quad (19)$$

In order to describe the anti-commutation relations of the dynamical supercharges it is useful to introduce $Q_{\dot{a}}^{\pm} = \frac{1}{\sqrt{2}}(Q_{\dot{a}} \pm i\tilde{Q}_{\dot{a}})$. Then the anti-commutation relations are [4] $\{Q_{\dot{a}}^{\pm}, Q_{\dot{b}}^{\pm}\} = 0$, as well as

$$\{Q_{\dot{a}}^+, Q_{\dot{b}}^-\} = 2\delta_{\dot{a}\dot{b}} H + m(\gamma^{ij} \Pi)_{\dot{a}\dot{b}} J^{ij} + m(\gamma^{i'j'} \Pi)_{\dot{a}\dot{b}} J^{i'j'}, \quad (20)$$

where J^{ij} are the rotation generators (see [4]) while H is the light-cone Hamiltonian H for the closed string in the plane-wave background

$$\begin{aligned} 2p^+ H &= m(a_0^I \bar{a}_0^I + i S_0^a \Pi_{ab} \tilde{S}_0^b + 4) + \sum_{k=1}^{\infty} [\alpha_{-k}^I \alpha_k^I + \tilde{\alpha}_{-k}^I \tilde{\alpha}_k^I + \omega_k (S_{-k}^a S_k^a + \tilde{S}_{-k}^a \tilde{S}_k^a)] \\ &= m(a_0^I \bar{a}_0^I + \theta_L^a \bar{\theta}_L^a + \bar{\theta}_R^a \theta_R^a) + \sum_{k=1}^{\infty} [\alpha_{-k}^I \alpha_k^I + \tilde{\alpha}_{-k}^I \tilde{\alpha}_k^I + \omega_k (S_{-k}^a S_k^a + \tilde{S}_{-k}^a \tilde{S}_k^a)]. \end{aligned} \quad (21)$$

In the limit $m \equiv 2\pi p^+ \mu \rightarrow 0$ this reduces to the usual light-cone gauge Hamiltonian in a flat background [23]. The normal ordering has been chosen in (21) with the understanding that θ_L^a and $\bar{\theta}_R^a$ are creation operators, while $\bar{\theta}_L^a$ and θ_R^a are annihilation operators.

It is easy to see from (21) that for $m \neq 0$, the ground state of the spectrum is a single bosonic state, but the excited states come in degenerate boson-fermion pairs. This is consistent with supersymmetry. In the plane-wave background, the kinematical supercharges do not commute with the light-cone Hamiltonian, but rather satisfy

$$[H, Q_a] = -\frac{im}{2p^+} \Pi_{ab} \tilde{Q}_b, \quad [H, \tilde{Q}_a] = \frac{im}{2p^+} \Pi_{ab} Q_b. \quad (22)$$

Since the kinematical supercharges do not commute with the Hamiltonian, we cannot use them to deduce (as one does in flat space) that the entire space must be fermion-boson degenerate. On the other hand, the dynamical supercharges do commute with the

²We are adopting a slightly different normalisation for the non-zero mode contributions.

light-cone Hamiltonian, but their anti-commutator (20) involves operators on the right-hand side that annihilate the bosonic ground state. Thus, as in the familiar situation with ‘unbroken supersymmetry’, the ground state is not boson-fermion degenerate, but all excited states are.

One can also show that the full closed string spectrum of the theory gives rise to a modular invariant torus amplitude [24, 25]. This is a consequence of modular identities that are very similar to those that will be discussed in the construction of D-branes below.

3 D-branes in the plane wave background

Given that we have an exact world-sheet description of the theory (at least in light-cone gauge) it is interesting to use it to give an exact world-sheet description of D-branes. In particular, we can describe D-branes in terms of (i) the open strings that begin and end on them; and (ii) in terms of the closed string boundary states that describe the coupling of the D-branes to the closed string excitations of the theory. Finally, we should expect that the results tie in with the corresponding supergravity analysis.

The boundary states are coherent states in the closed string theory that implement the effect of the corresponding D-brane. It is convenient to describe them in the usual light cone gauge in which $x^+ = 2\pi p^+ \tau$. The boundary states are then necessarily instantonic, *i.e.* they satisfy a Dirichlet boundary condition in the x^+ direction (as well as in x^-).³ In order to relate these boundary states to open strings one considers the cylinder diagram that describes the closed string exchange between two boundary states. This diagram also has a dual interpretation in terms of an open string 1-loop amplitude, and these two point of views are related to one another by exchanging the roles of the space and time-parameters on the world-sheet. [Since the length of the cylinder is ΔX^+ while its circumference is $2\pi p^+$, this transformation also exchanges the roles of ΔX^+ and $2\pi p^+$. As a consequence, the mass parameter of the resulting open string light cone gauge is $\widehat{m} = mt$, where $t = \Delta X^+ / 2\pi p^+$ is the modular parameter of the cylinder. This is explained in detail in [26, 27].] The open string 1-loop amplitude is simply a trace, and thus must be an integer linear combination of characters. This is the content of the so-called *open-closed string duality relation*.

A significant amount of work has been done on the construction of D-branes for the maximally supersymmetric plane-wave background. The open string point of view has been worked out (among others) in [28, 29, 30, 31, 27], while the boundary states were constructed in [22, 26, 27, 32, 33], and the open-closed duality relation relating the two was studied in [26, 27]. The supergravity analysis was performed, among others, in [29, 34, 35].

In the following we shall mainly discuss the half-supersymmetric D-branes. Before describing the results in detail, there is one basic point that should be stressed. As we have mentioned before, because of the fermion mass term, the background is actually only invariant under the $SO(4) \times SO(4)$ subgroup of the transverse $SO(8)$ symmetry. Unlike the situation in flat space, the D-branes are therefore not only characterised by the dimension of their world-volume, but also by the orientation of the world-volume relative to the background RR flux. [This is to say, while any two flat submanifolds of the same dimension can be mapped into one another by a rotation in $SO(8)$, this is in general not possible by a rotation in $SO(4) \times SO(4)$.]

³In order to construct boundary states for time-like branes one has to choose a different light-cone gauge; this is briefly discussed in [26]. Most properties of branes are however independent of whether the two light-cone directions are Dirichlet or Neumann.

For any D-brane of a given orientation, let us define

$$M = \prod_{I \in \mathcal{N}} \gamma^I, \quad (23)$$

where \mathcal{N} denotes an orthonormal basis for the Neumann directions of the brane. Then the nature of the brane will essentially only depend on the matrix [31, 27]

$$(\Pi M \Pi M)_{ab}. \quad (24)$$

This matrix encodes the relevant information about the orientation of the brane relative to the background RR flux. There are two cases to consider:

Class I: The first class is the one that was studied in [22, 28, 26] and arises when the matrix M_{ab} satisfies

$$(\Pi M \Pi M)_{ab} = -\delta_{ab}. \quad (25)$$

In this case the gluing conditions of the boundary state are the standard bosonic conditions

$$(\alpha_k^I \mp \tilde{\alpha}_{-k}^I) \|D, \eta\rangle = 0, \quad k \in \mathbb{Z}, \quad (26)$$

where for a Dirichlet (Neumann) direction I , the upper (lower) sign applies. The fermionic gluing conditions are simply [22]

$$(S_k^a + i\eta M_{ab} \tilde{S}_{-k}^b) \|D, \eta\rangle = 0, \quad k \in \mathbb{Z}, \quad (27)$$

where $\eta = \pm$ distinguishes a brane from an anti-brane. In terms of the open string boundary conditions this last identity becomes

$$S^a = \eta M_{ab} \tilde{S}^b, \quad \text{at } \sigma = 0, \pi, \quad (28)$$

together with the standard boundary conditions for the bosons [28, 29]. These branes preserve half the kinematical supersymmetries, and half the dynamical supersymmetries provided that the D-brane is at the origin in transverse space.⁴

If the brane is oriented in such a way that it has r Neumann directions among the first four (transverse) coordinates, as well as s Neumann directions among the second four (transverse) coordinates, it is useful to denote it as $D(r, s)$ [29]. It is not difficult to show that a $D(r, s)$ brane is of class I if $|r - s| = 2$. However, there also exist class I branes that are not of this type; the simplest example is the oblique D5-brane of [35] (*i.e.* a brane with four Neumann directions in transverse space) that was explicitly constructed in string theory in [33].

These branes are the obvious analogous of the usual flat space branes, and thus the open-closed duality relation should work as in flat space. There it is a consequence of the fact that the amplitudes can be expressed in terms of the functions [37]

$$\begin{aligned} f_1(q) &= q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \\ f_2(q) &= \sqrt{2} q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^n), \\ f_3(q) &= q^{-\frac{1}{48}} \prod_{n=1}^{\infty} (1 + q^{(n-1/2)}), \\ f_4(q) &= q^{-\frac{1}{48}} \prod_{n=1}^{\infty} (1 - q^{(n-1/2)}), \end{aligned} \quad (29)$$

⁴It was observed in [31, 32] that these D-branes preserve (modified) dynamical supersymmetries even if they are not at the origin in transverse space. The analysis of [36] however suggests that these modified supersymmetries are an artefact of the free theory, and do not survive interactions.

where $q = e^{-2\pi t}$. The duality transformation that relates the open and closed point of view is precisely the S-modular transformation under which $t \mapsto \tilde{t} = 1/t$. Writing $\tilde{q} = e^{-2\pi \tilde{t}}$, the above functions transform as

$$f_1(q) = t^{-\frac{1}{2}} f_1(\tilde{q}), \quad f_2(q) = f_4(\tilde{q}), \quad f_3(q) = f_3(\tilde{q}). \quad (30)$$

These (simple) transformation formulae guarantee that the cylinder amplitudes transform appropriately, and that the open-closed duality relation is satisfied. For class I branes, the relevant amplitudes involve instead of the f_i functions,

$$f_1^{(m)}(q) = q^{-\Delta_m} (1 - q^m)^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^{\sqrt{m^2+n^2}}), \quad (31)$$

$$f_2^{(m)}(q) = q^{-\Delta_m} (1 + q^m)^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 + q^{\sqrt{m^2+n^2}}), \quad (32)$$

$$f_3^{(m)}(q) = q^{-\Delta'_m} \prod_{n=1}^{\infty} (1 + q^{\sqrt{m^2+(n-1/2)^2}}), \quad (33)$$

$$f_4^{(m)}(q) = q^{-\Delta'_m} \prod_{n=1}^{\infty} (1 - q^{\sqrt{m^2+(n-1/2)^2}}), \quad (34)$$

where Δ_m and Δ'_m are given as

$$\begin{aligned} \Delta_m &= -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty} \int_0^{\infty} ds e^{-p^2 s} e^{-\pi^2 m^2/s}, \\ \Delta'_m &= -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty} (-1)^p \int_0^{\infty} ds e^{-p^2 s} e^{-\pi^2 m^2/s}. \end{aligned} \quad (35)$$

The quantities Δ_m and Δ'_m are the Casimir energies of a single (two-dimensional) boson of mass m on a cylindrical world-sheet with periodic and anti-periodic boundary conditions, respectively. For $m = 0$, Δ_m and Δ'_m simplify to the usual flat-space values,

$$\begin{aligned} \Delta_0 &= -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty} \frac{1}{p^2} = -\frac{1}{24}, \\ \Delta'_0 &= -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty} \frac{(-1)^p}{p^2} = \frac{1}{48}. \end{aligned} \quad (36)$$

Thus $f_2^{(m)}(q)$, $f_3^{(m)}(q)$ and $f_4^{(m)}(q)$ simply reduce to the standard $f_2(q)$, $f_3(q)$ and $f_4(q)$ functions [37] as $m \rightarrow 0$.

For the case of the class I branes the open-closed consistency condition is then a consequence of the remarkable transformation properties of these functions [26] (see also [38])

$$f_1^{(m)}(q) = f_1^{(\widehat{m})}(\tilde{q}), \quad f_2^{(m)}(q) = f_4^{(\widehat{m})}(\tilde{q}), \quad f_3^{(m)}(q) = f_3^{(\widehat{m})}(\tilde{q}), \quad (37)$$

where $\tilde{q} = e^{-2\pi \tilde{t}} = e^{-2\pi/t}$ and $\widehat{m} = mt$. In the limit $m \rightarrow 0$ the second and third equations in (37) reproduce the above identities for f_2 , f_3 and f_4 . The identity for f_1 (or η) can also be derived from the first equation of (37). In fact, both sides of the first equation tend to zero as $m \rightarrow 0$ since $(1 - q^m)^{\frac{1}{2}} = \sqrt{2\pi t m} + \mathcal{O}(m)$ and $(1 - \tilde{q}^{\widehat{m}})^{\frac{1}{2}} = \sqrt{2\pi \widehat{m}} + \mathcal{O}(m)$. Thus, after dividing the first equation by \sqrt{m} , the limit $m \rightarrow 0$ becomes

$$f_1(q) = t^{-\frac{1}{2}} f_1(\tilde{q}), \quad (38)$$

thus reproducing the standard modular transformation property of the f_1 (or η) function.

Class II: The second class arises when the matrix M_{ab} satisfies

$$(\Pi M \Pi M)_{ab} = \delta_{ab}, \quad (39)$$

a possibility that was not considered in [22, 28, 26] but arose in the supergravity analyses of [29, 34] and was later analysed in detail in [27] (the open string description was independently worked out in [31]). The only class II D-branes that preserve half the dynamical supersymmetries are the $(r, s) = (0, 0)$ brane and the $(r, s) = (4, 0)$ or $(r, s) = (0, 4)$ brane (provided that the Neumann boundary conditions for the bosons are modified). Neither of these branes preserves any kinematical supersymmetries, but now the dynamical supersymmetries are also preserved if the brane is moved away from the origin in transverse space. For the case of the D-instanton (or time-like D1-brane) the gluing conditions for the bosons are as above (all transverse directions are simply Dirichlet), but the fermionic gluing conditions are now

$$\begin{aligned} (S_0^a + i\eta \tilde{S}_0^a) \|(0, 0), \eta\rangle &= 0, \\ (S_n^a + i\eta R_n^{ab} \tilde{S}_{-n}^b) \|(0, 0), \eta\rangle &= 0, \end{aligned} \quad (40)$$

where R_n is the matrix

$$R_n = \frac{1}{n} (\omega_n \mathbb{1} - \eta m \Pi). \quad (41)$$

The corresponding open string boundary condition that corresponds to this gluing condition is simply

$$S = \eta \tilde{S}, \quad \text{at } \sigma = 0, \pi. \quad (42)$$

If we define the eigencomponents of Π by

$$\Pi S_n^\pm = \pm S_n^\pm \quad \Pi \tilde{S}_n^\pm = \pm \tilde{S}_n^\pm, \quad (43)$$

then the above gluing condition is simply

$$(S_n^\pm + i\eta R_n^{\pm} \tilde{S}_{-n}^\pm) \|(0, 0), \eta\rangle = 0, \quad (44)$$

where

$$R_n^\pm = \sqrt{\frac{\omega_n \mp \eta m}{\omega_n \pm \eta m}}. \quad (45)$$

It is then straightforward to calculate the cylinder amplitude involving two such boundary states, and one finds that it involves (if the two boundary states have opposite η) the function

$$g_2^{(m)}(q) = 4\pi m q^{-2\Delta_m} q^{m/2} \prod_{n=1}^{\infty} \left(1 + \left(\frac{\omega_n + m}{\omega_n - m}\right) q^{\omega_n}\right) \left(1 + \left(\frac{\omega_n - m}{\omega_n + m}\right) q^{\omega_n}\right). \quad (46)$$

Under the modular transformation $q \mapsto \tilde{q}$, this function becomes [27]

$$g_2^{(m)}(q) = \hat{g}_4^{(\hat{m})}(\tilde{q}), \quad (47)$$

where

$$\hat{g}_4^{(\hat{m})}(\tilde{q}) = \tilde{q}^{-\hat{\Delta}_{\hat{m}}} \prod_{l \in \mathcal{P}_+} (1 - \tilde{q}^{\hat{\omega}_l})^{1/2} \prod_{l \in \mathcal{P}_-} (1 - \tilde{q}^{\hat{\omega}_l})^{1/2}, \quad (48)$$

and $l \in \mathcal{P}_\pm$ provided that l satisfies the transcendental equation

$$\frac{l \pm i\widehat{m}}{l \mp i\widehat{m}} = -e^{2\pi i l}. \quad (49)$$

As is explained in detail in [27], this then reproduces precisely the open string 1-loop amplitude for an open string with boundary conditions (42) at the two ends (where the two values of η are different). As an aside we mention that there exist various generalisations of (47) that were also proven in [27].

There are also supersymmetric D-branes that are neither class I nor class II. For example the oblique D3-brane that was first discussed from a supergravity point of view in [35] corresponds to the choice

$$M_3 = \frac{1}{2}(\gamma^1 - \gamma^6)(\gamma^2 + \gamma^5), \quad (50)$$

and therefore satisfies

$$\Pi M_3 \Pi M_3 = \gamma^1 \gamma^2 \gamma^5 \gamma^6. \quad (51)$$

This matrix has four eigenvectors with eigenvalue $+1$, and four eigenvectors with eigenvalue -1 . The corresponding brane is therefore half way between being class I and class II. As is explained in detail in [33] it preserves a quarter of the kinematical and dynamical supersymmetries. The open-closed duality relations are satisfied by arguments that are a combination of the arguments that arise for class I and class II branes.

Finally, there are also supersymmetric D-branes whose world-volume is not flat. For example, there is a curved D7-brane whose world-volume is described by the equation [35]

$$\sum_{i=1}^4 Z_i Z_i = c, \quad Z_i = X_i + iX_{i+4}. \quad (52)$$

It was shown in [33] that the corresponding brane actually preserves two dynamical supersymmetries. In order to describe the relevant boundary conditions for the fermions in the open string we introduce the space-dependent matrix

$$M_7 = \frac{1}{x^I x^I} \left(x^i \gamma^i - x^{i+4} \gamma^{i+4} \right) \left(x^i \gamma^{i+4} + x^{i+4} \gamma^i \right). \quad (53)$$

The boundary condition for the fermions is then simply given by

$$S = \eta M_7 \tilde{S}, \quad \text{at } \sigma = 0, \pi. \quad (54)$$

The boundary condition for the bosons on the other hand receives a correction term due to the fact that M_7 depends on Z^i . In the above complex basis, the relevant boundary condition is then [33]

$$\left(P^{\bar{j}i} \partial_\sigma \bar{Z}^j - 2S \frac{\partial M_7}{\partial Z^i} \tilde{S} \right) = 0, \quad (55)$$

where $\bar{Z}^j = X^j - iX^{j+4}$ and

$$P^{\bar{j}l} = \left(\delta^{jl} - \frac{\bar{Z}^j Z^l}{|Z|^2} \right). \quad (56)$$

4 Conclusion

In this lecture we have given a brief overview over string theory in the maximally supersymmetric plane wave background. We have also described some of the supersymmetric D-branes of this background.

As has been mentioned before, the maximally supersymmetric plane-wave background is a background with non-trivial RR flux. It would be interesting to study such backgrounds further since they form a ‘corner’ of string theory that has been very little explored so far. The simplest generalisation are other plane-wave backgrounds for which the coefficient of the $(dx^+)^2$ term in the metric is of the form $A_{IJ}(x^+)x^I x^J$. The resulting world-sheet theories always preserve 16 kinematical supersymmetries since the fermions of these theories are always free. On the other hand, these backgrounds do not preserve in general any dynamical supersymmetries; if they do, this is a consequence of some of the bosonic fields being free as well. These backgrounds therefore only preserve dynamical supersymmetries if they behave in some respect as the maximally supersymmetric case.

A more interesting class of backgrounds are therefore pp-wave backgrounds for which the coefficient of $(dx^+)^2$ in the metric is not necessarily quadratic in the transverse coordinates. It was shown in [39] (see also [15]) that there are backgrounds of this type which preserve some dynamical supersymmetries without involving free boson and fermion fields. Furthermore it was shown that at least some of these do describe exact string solutions [40]. In the most interesting examples, the relevant world-sheet theories are integrable. It would be very interesting to make use of this integrable structure in order to gain insight into the nature of these theories. Some progress has recently been made in this direction in [41].

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